



## On $\alpha$ - Separation Axiom In Biminimal Space

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### ABSTRACT

This paper deals with the extension and development of some of the concepts of the separation axiom in topological space into biminimal spaces. In Addition, to give new results and theorems, supported by illustrative examples.

**KEYWORDS:** Biminimal space;  $M\alpha$ - $M_0$  space;  $M\alpha$ - $M_1$  space;  $M\alpha$ - $M_2$  space;  $M\alpha$ - $M_3$  space;  $M\alpha$ - $M_4$  space.



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## INTRODUCTION

Minimal spaces have been the subject of many research papers recently; see [1], [2], [3], [4], [5], [7] and [8]. Maki, H., Umehara J. and Noiri T. introduced the notions of minimal structure and minimal spaces in 1950, [9]. While the notion of biminimal space was introduced by Boonpok, C. [5] in 2010, which a set equipped with two minimal spaces is called a biminimal space, denoted by  $(X, M, N)$ , where  $(X, M), (X, N)$  are two minimal spaces defined on  $X$ .

In 2013, Hashoosh and Farawi gave definition of  $m\alpha$ -open in biminimal space. A subset  $A$  of  $X$  is said to be ( $m\alpha$ -open set in biminimal space) if  $A \subseteq M\text{-int} ( N - \text{cl} ( M\text{-int} ( A ) ) )$  [6].

In this paper we introduce new definitions of  $m\alpha$ - $M_0$  space,  $m\alpha$ - $M_1$  space and  $m\alpha$ - $M_2$  space, and give some theorems and results, for them in biminimal space. In addition to the definitions of  $m\alpha$ -regular space and  $m\alpha$ -normal space are introduced and studied. Moreover, We got a series of theorems and important results as well as the basic concepts and the relationships of it.

The aim of this paper is to continue the discussion new classes of separation axioms in biminimal spaces.

## 1. PRELIMINARIES

### 1.1. Definition [9]

Let  $X$  be a nonempty set. A family  $M \subseteq P(X)$  is said to be minimal structure on  $X$  if  $\emptyset, X \in M$ . In this case  $(X, M)$  is called an minimal space.

A set  $A \in P(X)$  is said to be an  $m$ -open if  $A \in M$ ,  $B \in P(X)$  is an  $m$ -closed set if  $B^c \in M$ . We set

$$m\text{-Int}(A) = \bigcup \{ U : U \subseteq A, U \in M \}$$

$$m\text{-Cl}(A) = \bigcap \{ F : A \subseteq F, F^c \in M \}.$$

### 1.2. Definition

i) Let  $(X, M)$  be an  $m$ -space then we say that  $(X, M)$  has the property  $\mathcal{U}$  if the arbitrary union of  $m$ -open sets is an  $m$ -open set [10].

ii) Let  $(X, M)$  be an  $m$ -space then we say that  $(X, M)$  has the property  $\mathcal{J}$  if the any finite intersection of an  $m$ -open sets is an  $m$ -open [1].

### 1.3. Definition [5]

Let  $(X, M), (X, N)$  are two minimal spaces defined on  $X$ , then the triple  $(X, M, N)$  is called a biminimal space.

### 1.4. Definition [7].

Let  $A$  be a subset of  $X$ , then  $A$  is said to be  $m\alpha$ -open set in biminimal space iff  $A \subseteq M\text{-int} ( N - \text{cl} ( M\text{-int} ( A ) ) )$ . The family of all  $m\alpha$ -open set of  $X$  is denoted by  $m\alpha(X)$ .

### 1.5. Example:

Let  $X = \{ a, b, c \}$ ,  $M = \{ X, \emptyset, \{a\}, \{b\} \}$  and  $N = \{ X, \emptyset, \{a\}, \{c\} \}$

$(X, M), (X, N)$  are two minimal spaces on  $X$ , then  $(X, M, N)$  is a biminimal space.

$$m\alpha(X) = \{ X, \emptyset, \{a\}, \{b\}, \{a, b\} \}.$$

Assume  $Y = \{a\}$ , then  $M\text{-int}(\{a\}) = \{a\}$  and  $N\text{-cl}(M\text{-int}\{a\}) = \{b, a\}$  hence  $M\text{-int}(N\text{-cl}(M\text{-int}\{a\})) = M\text{-int}(\{a, b\}) = \{a, b\}$ . Thus,  $\{a\} \subseteq M\text{-int}(N\text{-cl}(M\text{-int}\{a\}))$ . Therefore  $\{a\}$  is  $m\alpha$ -open set in  $(X, M, N)$ , and in general in any biminimal space  $X$ , both  $X$  and  $\emptyset$  are clearly  $m\alpha$ -open sets, so are the other cases  $\{b\}, \{a, b\}$ .

### 1.6. Remarks [7]

Let  $(X, M, N)$  be a biminimal space.

(1) A subset  $Y$  of  $X$  is called  $m\alpha$ -closed set of  $X$  if the complement of  $Y$  is  $m\alpha$ -open set of  $X$ .

(2) Every  $M$ -open set is  $m\alpha$ -open set.

(3)  $m\alpha\text{-cl}\{x\} \subseteq m\alpha\text{-cl}(m\alpha\text{-cl}\{x\})$ .

## 2. $M\alpha$ - $M_0$ SPACE, $M\alpha$ - $M_1$ SPACE AND $M\alpha$ - $M_2$ SPACE IN BIMINIMAL SPACE

In this section, we introduce new definitions of  $m\alpha$ - $M_0$  space,  $m\alpha$ - $M_1$  space and  $m\alpha$ - $M_2$  space, in addition to we recall some the results and relations between them with illustrated examples.

### 2.1. Definition

Let  $(X, M, N)$  be a biminimal space, then  $(X, M, N)$  is called  $m\alpha$ - $M_0$  space iff for each pair of points  $x, y$  of  $X$ , such that  $x \neq y$ , there exists  $m\alpha$ -open set  $G$  containing  $x$  but not containing  $y$  or  $m\alpha$ -open set  $H$  containing  $y$  but not containing  $x$ .



## 2.2. Example

Let  $X = \{a, b, c\}$ ,  $M = \{X, \emptyset, \{a\}, \{b\}\}$  and  $N = \{X, \emptyset, \{a\}, \{c\}\}$

$(X, M)$  and  $(X, N)$  are two minimal spaces on  $X$ , then  $(X, M, N)$  is a biminimal space and  $\text{ma}(X) = \{X, \emptyset, \{a\}, \{b\}, \{a, b\}\}$ .

We note for every  $\alpha$  and  $\beta$  in  $X$  such that  $\alpha \neq \beta$ , there exists  $\text{ma}$ -open  $\{a\}$  contains  $\alpha$  but not containing  $\beta$ , therefore then  $(X, M, N)$  is  $\text{ma}$ - $M_0$  space.

## 2.3. Proposition

Let  $(X, M, N)$  be a biminimal space, if  $(X, M)$  is  $M_0$  space, then  $(X, M, N)$  is  $\text{ma}$ - $M_0$  space.

### Proof:

Let  $x, y \in X$  such that  $x \neq y$ . Since  $(X, M)$  is  $M_0$ -space, there exists  $M$ -open set  $A$  in  $X$  such that  $x \in A$  and  $y \notin A$ .

Since every  $M$ -open set is  $\text{ma}$ -open set by (1.6) Remarks, so  $A$  is  $\text{ma}$ -open set such that  $x \in A$  and  $y \notin A$ . Therefore  $(X, M, N)$  is  $\text{ma}$ - $M_0$  space. But the opposite of this proposition is not true (see 2.4. example below).

## 2.4. Example

Let  $X = \{a, b, c\}$ ,  $M = \{X, \emptyset, \{a\}, \{b, c\}\}$  and  $N = \{X, \emptyset, \{a\}\}$ . Then,  $(X, M)$  is not  $M_0$ -space. Since,  $\text{ma}(X) = \{X, \emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{b, c\}, \{a, c\}\}$ , so  $(X, M, N)$  is  $\text{ma}$ - $M_0$  space.

## 2.5. Theorem

Let  $(X, M, N)$  be biminimal space with property  $\mathcal{J}$ , then  $(X, M, N)$  is  $M_0$  space iff for each distinct point  $x, y$  of  $X$ ,  $\text{ma-cl}\{x\} \neq \text{ma-cl}\{y\}$ .

### Proof :

Let  $x, y \in X$ , such that  $x \neq y$  and let  $\text{ma-cl}\{x\} \neq \text{ma-cl}\{y\}$ . Then there exists at the least one point  $w$  in  $X$  such that  $w \in \text{ma-cl}\{x\}$ ,  $w \notin \text{ma-cl}\{y\}$ . Suppose that  $x \in \text{ma-cl}\{y\}$ , so  $\{x\} \subseteq \text{ma-cl}\{y\}$ . Then  $\text{ma-cl}\{x\} \subseteq \text{ma-cl}(\text{ma-cl}\{y\})$ , by (1.6) Remarks.

But  $\text{ma-cl}(\text{ma-cl}\{y\}) = \text{ma-cl}\{y\}$ , then  $w \in \text{ma-cl}\{y\}$  and this is a contradiction.

So  $x \notin \text{ma-cl}\{y\}$ , then  $x \in X - (\text{ma-cl}\{y\})$ . Since  $\text{ma-cl}\{y\}$  is  $\text{ma}$ -closed set, so  $X - (\text{ma-cl}\{y\})$  is  $\text{ma}$ -open set.

Therefore,  $X - (\text{ma-cl}\{y\})$  is  $\text{ma}$ -open set containing  $x$  but not  $y$ . Thus  $(X, M, N)$  is  $\text{ma}$ - $M_0$  space. Conversely, since  $(X, M, N)$  is  $\text{ma}$ - $M_0$  space, then for each two distinct point  $x, y \in X$ , there exists a  $\text{ma}$ -open set  $G$  such that  $x \in G$ ,  $y \notin G$ .  $X - G$  is  $\text{ma}$ -closed set which does not contain  $x$ , but contains  $y$ . By definition (2.1),  $\text{ma-cl}\{y\}$  is the intersection of all  $\text{ma}$ -closed set containing  $\{y\}$ . Thus  $\text{ma-cl}\{y\} \subseteq X - G$ , then  $x \notin X - G$ . This implies that  $x \notin \text{ma-cl}\{y\}$ , so we have  $x \in \text{ma-cl}\{x\}$ ,  $x \notin \text{ma-cl}\{y\}$ .

Therefore  $\text{ma-cl}\{x\} \neq \text{ma-cl}\{y\}$ .

## 2.6. Theorem

The  $\text{ma}$ - $M_0$  space has the hereditary property, if  $\text{ma}$ - $M_0$  space has the property  $\mathcal{J}$ .

### Proof :

Let  $Y$  be a minimal subspace of  $\text{ma}$ - $M_0$  space  $X$ , to prove  $Y$  is  $\text{ma}$ - $M_0$  space, let  $z_1 \neq z_2 \in Y$ , since  $Y \subseteq X$ .

Then  $z_1 \neq z_2 \in X$  and  $X$  is  $\text{ma}$ - $M_0$  space. There exists  $\text{ma}$ -open set  $G$  in  $X$ , such that  $z_1 \in G$  and  $z_2 \notin G$ , so  $G \cap Y$  is  $\text{ma}$ -open set in  $Y$ , and  $z_1 \in G \cap Y$  and  $z_2 \notin G \cap Y$ .

Thus,  $Y$  is a  $\text{ma}$ - $M_0$  space.

## 2.7. Definition

A biminimal space  $(X, M, N)$  is called  $\text{ma}$ - $M_1$  space iff for each pair of distinct points  $x, y$  of  $X$  there exists two  $\text{ma}$ -open sets  $U, V$  such that  $x \in U$ ,  $y \notin U$  and  $y \in V$ ,  $x \notin V$ .

## 2.8. Theorem

If  $(X, M)$  is  $M_1$  space, then  $(X, M, N)$  is  $\text{ma}$ - $M_1$  space.

### Proof:

Let  $a, b \in X$ ,  $a \neq b$ . Since  $(X, M)$  is  $M_1$  space, then there exists two  $M$ -open sets  $A$  and  $B$  in  $X$  such that  $a \in A$ , but  $b \notin A$ , and  $b \in B$ , but  $a \notin B$ . Since every  $M$ -open set is  $\text{ma}$ -open set by (1.6) Remarks, then  $A, B$  are  $\text{ma}$ -open sets, such that  $a \in A$ , but  $b \notin A$ ; and  $b \in B$ , but  $a \notin B$ . Therefore  $(X, M, N)$  is  $\text{ma}$ - $M_1$  space.

## 2.9. Remark



The converse of the above proposition is not true, that is if  $(X, M, N)$  is  $\alpha$ - $M_1$  space, then it is not true that  $(X, M)$  is  $M_1$  space by the previous example.

### 2.10. Theorem

The  $\alpha$ - $M_1$  space has the hereditary property, such that  $\alpha$ - $M_1$  space has the property  $\mathcal{J}$ .

**Proof:**

Let  $(X, M, N)$  be a  $\alpha$ - $M_1$  space and let  $(Y, M_Y, N_Y)$  be a minimal sub space of  $(X, M, N)$ , and assume  $y_1 \neq y_2 \in Y$  and since  $Y \subseteq X$ , then  $y_1 \neq y_2 \in X$ .

Since  $X$  is  $\alpha$ - $M_1$  space, there exist two  $\alpha$ -open sets  $U, V$  in  $X$ , such that  $y_1 \in U$ , but  $y_2 \notin U$ , and  $y_1 \in V$ , but  $y_2 \notin V$ . Then  $U_1 = U \cap Y$ ,  $V_1 = V \cap Y$  are  $\alpha$ -open sets in  $Y$  and we have  $y_1 \in U_1$ , but  $y_2 \notin U_1$ ; and  $y_1 \in V_1$ , but  $y_2 \notin V_1$ . Therefore  $(Y, M_Y, N_Y)$  is  $\alpha$ - $M_1$  space.

### 2.11. Theorem

A biminimal space  $(X, M, N)$  is a  $\alpha$ - $M_1$  space iff every single subset  $\{x\}$  of  $X$  is  $\alpha$ -closed.

**Proof :**

Suppose  $X$  is  $\alpha$ - $M_1$  space, and  $x$  be any point of  $X$ .

Let  $y \in \{x\}^c$ , then  $x \neq y$  and so there exists  $\alpha$ -open set  $U$  containing  $y$  but not  $x$ , and  $\alpha$ -open set  $V$  containing  $x$  but not containing  $y$ ,  $y \in U \subseteq \{x\}^c$ . Therefore  $\{x\}^c$  is  $\alpha$ -open set, then  $\{x\}$  is  $\alpha$ -closed set. Conversely, suppose  $x, y \in X$ , such that  $x \neq y$ . Since  $\{x\}$  is  $\alpha$ -closed set, then  $\{x\}^c$  is  $\alpha$ -open set containing  $y$  but not  $x$ . Similarly,  $\{y\}^c$  is  $\alpha$ -open set containing  $x$  but not containing  $y$ . Therefore  $(X, M, N)$  is  $\alpha$ - $M_1$  space.

### 2.12. Theorem

Let  $(X, M, N)$  be a biminimal space has the property  $\mathcal{J}$ , then  $(X, M, N)$  is  $\alpha$ - $M_1$  space iff  $\alpha$ -cl $\{a\}$  is empty set for each  $a \in X$

**Proof:**

Suppose  $(X, M, N)$  be a  $\alpha$ - $M_1$  space. Let  $\alpha$ -cl $\{a\} \neq \emptyset$ , for some  $a \in X$ , then there is a point  $b$ , such that  $b \in \alpha$ -cl $\{a\}$ , and  $b \neq a$ . Since  $X$  is  $\alpha$ - $M_1$  space, then there exist  $\alpha$ -open set  $G$  such that  $a \notin G$ ,  $b \in G$ . Thus  $G \cap \{a\} = \emptyset$ .

Therefore  $b \notin \alpha$ -cl $\{a\}$ , which is contradiction. Thus  $\alpha$ -cl $\{a\}$  is empty set.

On the other hand, Let  $\alpha$ -cl $\{a\}$  is empty set, for each  $a \in X$ , and let  $x, y \in X$ , such that  $x \neq y$ . Then  $x \notin \alpha$ -cl $\{y\}$ , and there exists  $\alpha$ -open set  $G$  such that  $x \in G$  and  $G \cap \{y\} = \emptyset$ , Therefore  $G$  contains  $x$  but not containing  $y$ . Similarly, there exists  $\alpha$ -open set contains  $y$  but not containing  $x$ . Thus  $(X, M, N)$  is  $\alpha$ - $M_1$  space.

### 2.13. Definition

Let  $(X, M, N)$  be a biminimal space has the property  $\mathcal{J}$ , then  $(X, M, N)$  is called  $\alpha$ - $M_2$  space ( $\alpha$ -Hausdorff) iff for each pair of distinct points  $x, y$  of  $X$  there exists two  $\alpha$ -open sets  $G, H$  Such that  $x \in G, y \in H, G \cap H = \emptyset$ .

### 2.14. Proposition

Let  $(X, M, N)$  be a biminimal space has the property  $\mathcal{J}$ . If  $(X, M)$  is  $M_2$  space, then  $(X, M, N)$  is  $\alpha$ - $M_2$  space.

**Proof :**

Assume  $x, y \in X$ , such that  $x \neq y$ . Since  $(X, M)$  is  $M_2$  space, then there exist two  $M$ -open set  $U$  and  $V$  in  $X$  such that  $x \in U, y \in V$  and  $U \cap V = \emptyset$ .

Since every  $M$ -open set is  $\alpha$ -open by (1.6) Remarks, then  $U, V$  are  $\alpha$ -open sets such that  $x \in U, y \in V, U \cap V = \emptyset$ . Hence  $(X, M, N)$  is  $\alpha$ - $M_2$  space. The opposite of this proposition is not correct by the following example.

### 2.15. Example

Let  $(X, M, N)$  be biminimal space has the property  $\mathcal{J}$  such that

$X = \{a, b, c\}$ ,  $M = \{X, \emptyset, \{a\}, \{b\}, \{a, c\}$  and  $N = \{X, \emptyset, (X, M), (X, N)\}$  are two minimal spaces on  $X$ , then  $(X, M, N)$  is a biminimal space. Then

$\alpha.o(X) = \{X, \emptyset, \{a\}, \{b\}, \{a, b\}, \{a, c\}, \{b, c\}\}$ .

We note  $(X, M)$  is not  $M_2$ -space, while  $(X, M, N)$  is  $\alpha$ - $M_2$  space.

### 2.16. Remark

Every  $\alpha$ - $M_2$  space is a  $\alpha$ - $M_1$  space, but the opposite is not correct (see 2.17. example below).

### 2.17. Example



Let  $X = \{a, b, c\}$ ,  $M = \{X, \emptyset, \{b\}, \{a, b\}, \{b, c\}\}$  and  $N = \{X, \emptyset, \{a, b\}, (X, M), (X, N)\}$  are two minimal spaces on  $X$ , then  $(X, M, N)$  is a biminimal space such that, it has the property  $\mathcal{J}$ . Then  $\text{m}\alpha\text{-o}(X) = \{X, \emptyset, \{b\}, \{a, b\}, \{b, c\}, \{a, c\}\}$ . So,  $(X, M, N)$  is a  $\text{m}\alpha\text{-M}_1$  space, but is not a  $\text{m}\alpha\text{-M}_2$  space.

**2.18. Theorem**

The  $\text{m}\alpha\text{-M}_2$  space has the hereditary property, such that  $\text{m}\alpha\text{-M}_2$  space has the property  $\mathcal{J}$ .

**Proof :**

Let  $(X, M, N)$  be a  $\text{m}\alpha\text{-Hansdorff}$ , such that  $\emptyset \neq Y \subseteq X$ , and  $x \neq y \in Y$ , then  $x \neq y \in X$  and, since  $(X, M, N)$  is  $\text{m}\alpha\text{-Hansdorff}$ , there exists two  $\text{m}\alpha\text{-open}$  sets  $G, H$  such that  $x \in G, y \in H, G \cap H = \emptyset$ .

So  $G \cap Y, H \cap Y$ , is  $\text{m}\alpha\text{-open}$  set in  $Y$ , and  $x \in G \cap Y, y \in H \cap Y$ , and  $(G \cap Y) \cap (H \cap Y) = (G \cap H) \cap Y = \emptyset$ . Therefore  $(Y, M_Y, N_Y)$  is  $\text{m}\alpha\text{-M}_2$  space.

**3. ON  $\text{M}\alpha\text{-REGULAR SPACE AND  $\text{M}\alpha\text{-NORMAL SPACE}$$**

In this section, we introduce new definitions of  $\text{m}\alpha\text{-M}_3$  space,  $\text{m}\alpha\text{-M}_4$  space,  $\text{m}\alpha\text{-regular}$  and  $\text{m}\alpha\text{-normal}$  space, as well as, we recall several the results and facts between them.

**3.1. Definition**

Let  $(X, M, N)$  be biminimal space has the property  $\mathcal{J}$ , then  $(X, M, N)$  is called  $\text{m}\alpha\text{-regular}$  space iff for each  $\text{m}\alpha\text{-closed}$  set  $F$  is in  $X$ , and each  $x \notin F$ ; and there exist  $\text{m}\alpha\text{-open}$  sets  $U, V$  such that  $x \in U, F \subseteq V, U \cap V = \emptyset$ .

**3.2. Proposition**

Let  $(X, M, N)$  be biminimal space has the property  $\mathcal{J}$ , then if  $(X, M)$  is regular space, then  $(X, M, N)$  is  $\text{m}\alpha\text{-regular}$ .

**Proof :**

Let  $F$  be  $M\text{-closed}$  set in  $X, x \in X$  such that  $x \notin F$ . Since  $(X, M)$  is regular space, then there exists  $G, H$  are  $M\text{-open}$  set in  $X$  such that  $x \in G, F \subseteq H, G \cap H = \emptyset$ . Since every  $M\text{-open}$  set is  $\text{m}\alpha\text{-open}$  set by ((1.6) Remarks), then  $G, H$  are  $\text{m}\alpha\text{-open}$  sets, such that  $x \in G, F \subseteq H, G \cap H = \emptyset$ . Hence  $(X, M, N)$  is  $\text{m}\alpha\text{-regular}$ .

**3.3. Remark**

The converse of the above proposition is not true as shown in the following example.

**3.4. Example**

Let  $(X, M, N)$  be a biminimal space, such that :

$X = \{a, b, c\}$ ,

$M = \{X, \emptyset, \{a\}, \{b\}, \{c\}, \{a, c\}, \{a, b\}\}$ ,

$N = \{X, \emptyset, \{c\}\}$ .

$(X, M), (X, N)$  are two  $m\text{-spaces}$  on  $X$ .

Then  $\text{m}\alpha\text{-o}(X) = \{X, \emptyset, \{a\}, \{b\}, \{c\}, \{b, c\}, \{a, c\}, \{a, b\}\}$ .

Take  $K = \{b, c\}, a \notin K$ , then there exists  $\{a\}, \{b, c\}$

$\text{m}\alpha\text{-open}$  sets such that  $a \in \{a\}, \{b, c\} \subseteq \{b, c\}$ ,

and  $\{a\} \cap \{b, c\} = \emptyset$ . And similarly the other cases. Hence  $(X, M, N)$  is  $\text{m}\alpha\text{-regular}$  space, but  $(X, M)$  is not  $m\text{-regular}$  space.

**3.5. Theorem**

Let  $(X, M, N)$  be biminimal space has the property  $\mathcal{J}$ , then  $(X, M, N)$  is  $\text{m}\alpha\text{-regular}$  iff for each  $\text{m}\alpha\text{-open}$  set  $U$  and  $x \in U$ , there exists  $\text{m}\alpha\text{-open}$  set  $V$  such that  $x \in V, \text{m}\alpha\text{-cl}(V) \subseteq U$ .

**Proof :**

Let  $(X, M, N)$  be  $\text{m}\alpha\text{-regular}$  space. Let  $x \in U$  where  $U$  is  $\text{m}\alpha\text{-open}$ . Let  $H = U^c$ , then  $H$  is  $\text{m}\alpha\text{-closed}$ ,  $x \notin H$ . Hence there exists  $\text{m}\alpha\text{-open}$  sets  $K$  and  $M$  such that :  $x \in M, H \subseteq K, M \cap K = \emptyset$ .

Then  $M \subseteq K^c, \text{m}\alpha\text{-cl}(M) \subseteq \text{m}\alpha\text{-cl}(K^c) = K^c$  .....(1)

$H \subseteq K$ , then  $K^c \subseteq H^c = U$ , then  $K^c \subseteq U$  .....(2)

From (1), (2) we have,  $x \in M, \text{m}\alpha\text{-cl}(M) \subseteq U$ .

Conversely :



Let  $H$  be  $m\alpha$ -closed set and  $x \notin H$ . Let  $U = H^c$ , then  $U$  is  $m\alpha$ -open and  $x \in U$ . By hypothesis, there exists  $m\alpha$ -open set  $M$  such that  $x \in M$ ,  $m\alpha\text{-cl}(M) \subseteq U$ ,  $H \subseteq (m\alpha\text{-cl}(M))^c$ . Since  $x \in M$ ,  $M \cap (m\alpha\text{-cl}(M))^c = \emptyset$ . Hence  $(X, M, N)$  is  $m\alpha$ -regular.

### 3.6. Theorem

The  $m\alpha$ -regular space has the hereditary property, such that  $m\alpha$ -regular space has the property  $\mathcal{J}$ .

#### Proof :

Assume  $(X, M, N)$  is  $m\alpha$ -regular space, let  $(Y, M_y, N_y)$  be a subspace of  $X$ . To prove  $(Y, M_y, N_y)$  is  $m\alpha$ -regular, Let  $q \in Y$  and  $F$  be  $m\alpha$ -closed set in  $Y$ , such that  $q \notin F$ . Then  $m\alpha\text{-cl}_Y(F) = m\alpha\text{-cl}_X(F) \cap Y$ , and since  $F$  is  $m\alpha$ -closed in  $Y$  so  $m\alpha\text{-cl}_Y(F) = F$ . Then  $F = m\alpha\text{-cl}_X(F) \cap Y$ . Since  $q \notin F$ , then  $q \notin m\alpha\text{-cl}_X(F) \cap Y$ ,  $q \notin m\alpha\text{-cl}_X(F)$ , thus  $m\alpha\text{-cl}_X(F)$  is  $m\alpha$ -closed in  $X$ , and since  $(X, M, N)$  is  $m\alpha$ -regular, then there exist two disjoint  $m\alpha$ -open sets  $G, H$  in  $X$ , such that  $q \in G$ ,  $m\alpha\text{-cl}_X(F) \subseteq H$  and  $G \cap H = \emptyset$ . Hence  $q \in G \cap Y$  and  $m\alpha\text{-cl}_X(F) \cap Y \subseteq H \cap Y$ ,  $F \subseteq H \cap Y$ , since  $G, H$  are  $m\alpha$ -open in  $X$  then  $G \cap Y, H \cap Y$  are  $m\alpha$ -open set in  $Y$ . Since  $G \cap H = \emptyset$ , then  $(G \cap Y) \cap (H \cap Y) = (G \cap H) \cap Y = \emptyset \cap Y = \emptyset$ . Therefore  $(Y, M_y, N_y)$  is  $m\alpha$ -regular subspace of  $(X, M, N)$ .

### 3.7. Definition

Let  $(X, M, N)$  be biminimal space has the property  $\mathcal{J}$ , then  $(X, M, N)$  is called  $m\alpha$ -normal space iff for each pair of  $m\alpha$ -closed set  $G, H$  in  $X$ , such that  $G \cap H = \emptyset$ , there exists  $m\alpha$ -open sets  $U, V$  such that  $G \subseteq U, H \subseteq V$  and  $U \cap V = \emptyset$ .

### 3.8. Proposition

Let  $(X, M, N)$  be biminimal space has the property  $\mathcal{J}$ , then  $(X, M, N)$  is  $m\alpha$ -normal space if  $(X, T)$  is normal space.

Let  $R, S$  be two  $M$ -closed set in  $X$ , such that  $R \cap S = \emptyset$ . Since  $(X, M)$  is normal space, then there exists  $G, H$  are  $M$ -open set in  $X$  such that  $S \subseteq G, R \subseteq H, G \cap H = \emptyset$ , but every  $M$ -open set is  $m\alpha$ -open set by ((1.6) Remarks), then  $G, H$  are  $m\alpha$ -open sets, such that  $S \subseteq G, R \subseteq H$  and  $G \cap H = \emptyset$ . Hence  $(X, M, N)$  is  $m\alpha$ -regular.

### 3.9. Remark

The converse of the above proposition is not correct, as shown in the following example.

#### 3.10. Example

Let  $(X, M, N)$  be biminimal space has the property  $\mathcal{I}$ , such that

$$X = \{a, b, c, d\},$$

$$M = \{\emptyset, X, \{a\}, \{a, d\}, \{b, c\}, \{c, d\}, \{a, b, c\}\},$$

$$N = \{X, \emptyset\}, \text{ such that :}$$

$$m\alpha\text{-o}(X) = \{\emptyset, X, \{a\}, \{a, d\}, \{b, c\}, \{c, d\}, \{a, b, c\}, \{a, b\}, \{a, c\}, \{a, c, d\}, \{b, c, d\}, \{a, b, d\}\}.$$

Then  $(X, M, N)$  is  $m\alpha$ -normal space, but  $(X, M)$  is not  $m$ -normal space.

### 3.11. Theorem

Let  $(X, M, N)$  be biminimal space has the property  $\mathcal{J}$ . Then  $(X, M, N)$  is  $m\alpha$ -normal space iff for every  $m\alpha$ -closed set  $H$  in  $X$  and  $m\alpha$ -open set  $U$  in  $X$  containing  $H$ , there exist  $m\alpha$ -open set  $V$ , such that  $H \subseteq V \subseteq m\alpha\text{-cl}(V) \subseteq U$ .

#### Proof :

Suppose  $(X, M, N)$  is  $m\alpha$ -normal space, let  $H$  be  $m\alpha$ -closed in  $X$  and  $U$  is  $m\alpha$ -open in  $X$ , such that  $H \subseteq U$ . Then  $U^c$  is  $m\alpha$ -closed in  $X$  and  $H \cap U^c = \emptyset$ . So there exist  $m\alpha$ -open sets  $V, K$  such that  $U^c \subseteq K, H \subseteq V, V \cap K = \emptyset, K^c \subseteq U, V \subseteq K^c$ . This implies that  $m\alpha\text{-cl}(V) \subseteq m\alpha\text{-cl}(K^c) = K^c$ . Then  $H \subseteq V \subseteq m\alpha\text{-cl}(V) \subseteq U$ .

Conversely, let  $H$  and  $G$  be  $m\alpha$ -closed sets in  $X$  such that  $H \cap G = \emptyset$ , then  $G^c$  is  $m\alpha$ -open in  $X$ , and  $H \subseteq G^c$ . By hypothesis, there exist  $m\alpha$ -open set  $V$ , such that  $H \subseteq V, m\alpha\text{-cl}(V) \subseteq G^c$ , then  $G \subseteq (m\alpha\text{-cl}(V))^c$ . So we have  $H \subseteq V, G \subseteq (m\alpha\text{-cl}(V))^c$ , and  $V \cap (m\alpha\text{-cl}(V))^c = \emptyset$ . Therefore,  $(X, M, N)$  is  $m\alpha$ -normal space.

### 3.12. Corollary

Let  $(X, M, N)$  be biminimal space has the property  $\mathcal{J}$ , then  $(X, M, N)$  is  $m\alpha$ -normal space iff for each  $m\alpha$ -closed set  $H$  in  $X$  and each  $m\alpha$ -open set  $U$  in  $X$  containing  $H$ , there exists a subset  $A$  of  $X$ , such that  $H \subseteq m\alpha\text{-int}(A) \subseteq m\alpha\text{-cl}(A) \subseteq U$ .

#### Proof:

To prove 3.12. Corollary, we only replace the  $m\alpha$ -open set  $V$  in 3.11. Theorem by a subset  $A$  of  $X$  with observing  $m\alpha\text{-int}(V) = V$ . Thus, we have finished the proof.

### 3.13. Definition

Let  $(X, M, N)$  be biminimal space has the property  $\mathcal{J}$ , then  $(X, M, N)$  is called a  $m\alpha$ - $M_3$  space iff  $X$  is  $m\alpha$ - $M_1$  and  $m\alpha$ -regular.



### 3.14. Remark

Every  $\alpha$ - $M_3$  space is  $\alpha$ -regular and the converse is not true in general (see 3.15. example below).

### 3.15. Example

Let  $X = \{a, b, c\}$ ,  $M = \{\emptyset, X, \{b\}, \{a, c\}\}$ ,  $N = \{\emptyset, X, \{c\}\}$ , then  $\alpha$ - $\alpha(X) = \{\emptyset, X, \{b\}, \{a, c\}\}$ . So, we have  $(X, M, N)$  is  $\alpha$ -regular space, but it is not  $\alpha$ - $M_1$  space, therefore it is not  $\alpha$ - $M_3$  space.

### 3.16. Definition

Let  $(X, M, N)$  be a biminimal space has the property  $\mathcal{J}$ , then  $(X, M, N)$  is called  $\alpha$ - $M_4$  space iff  $X$  is  $\alpha$ -normal and  $\alpha$ - $M_1$  space.

### 3.17. Remark

Every  $\alpha$ - $M_4$  space is  $\alpha$ -normal and the converse is not correct in general, (see in 3.15. Example,  $(X, M, N)$  is also  $\alpha$ -normal space but it is not  $\alpha$ - $M_1$  space, therefore it is not  $\alpha$ - $M_4$  space).

### 3.18. Proposition

Every  $\alpha$ - $M_4$  space is also  $\alpha$ - $M_3$  space.

#### Proof :

Let  $(X, M, N)$  be a  $\alpha$ - $M_4$  space, then  $(X, M, N)$  is  $\alpha$ -normal as well as  $\alpha$ - $M_1$  space. To prove that the space is  $\alpha$ - $M_3$  space, it suffices to show that the space is  $\alpha$ -regular.

Let  $F$  be a  $\alpha$ -closed subset of  $X$  and, let  $x$  be a point of  $X$  such that  $x \notin F$ . Since  $(X, M, N)$  is a  $\alpha$ - $M_1$  space. Thus  $\{x\}$  is a  $\alpha$ -closed subset of  $X$ , such that  $\{x\} \cap F = \emptyset$ , then by  $\alpha$ -normality, there exist  $\alpha$ -open sets disjoint  $G, H$  such that  $\{x\} \subseteq G, F \subseteq H$ . Also  $\{x\} \subseteq G$ , then  $x \in G$ , then there exist  $\alpha$ -open sets  $G, H$  such that  $x \in G, F \subseteq H$  and  $G \cap H = \emptyset$ . It follows that the space  $(X, M, N)$  is  $\alpha$ -regular.

## 5. REFERENCES

- [1] Alimohammady, M. and Roohi M., Fixed Point in Minimal Spaces, Modelling and Control, Vol. 10, No. 4, 305-314, (2005).
- [2] Alimohammady, M. and Roohi M., Delavar MR. Knaster-Kuratowski-Mazurkiewicz theorem in minimal generalized convex spaces, Nonlinear Funct Anal Appl, in press.
- [3] Alimohammady, M. and Roohi M., Extreme points in minimal spaces. Chaos, Solitons & Fractals ;39(3):1480-5,(2009).
- [4] Alimohammady, M. and Roohi M., Linear minimal space. Chaos, Solitons & Fractals, 33(4):1348-54,(2007).
- [5] Boonpok, C., Biminimal structure spaces, Int. Math. Forum, 5(15),703-707. (2010).
- [6] Hashoosh, A .E. and Farawi Y.A. ,  $M\delta$ -closed set in  $M-T_0$ -Alexandroff space. Journal of College of Education for pure science , vol.3 (1) ,(2013).
- [7] Hashoosh, A .E., PRE-ALEXANDROFF SPACE, J.Thi-Qar Sci. Vol.3 (4), 2013.
- [8] Hashoosh, A .E. and Farawi Y.A. ,  $M\theta$ -closed set in  $M-T_0$ -Alexandroff space. Preprint, Journal of University of thi-qar , accepted (2014).
- [9] Maki, H., Umehara J. and Noiri T. , Every topological space is pre  $T_{1/2}$  , Mem. Fac. Sci. Kochi Univ. Ser. Math, PP. 33-42 , (1950).
- [10] Sharmil, A.S. and Arockia R. , "generalized preregular closed sets in minimal structure space.international journal of computer application , issue ,volume 5, 2250-1797. (2012).



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