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SOME NEW PROPERTIES ON TOPOLOGICAL SPACES

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Abstract

In this journey, we are going to explore the "separation axioms" in greater detail. Separation axioms are one among the most common, important and interesting concepts in Topology. They can be used to define more restricted classes of topological spaces. We shall try to understand how these axioms are affected on subspaces, taking products, and looking at small open neighborhoods.

Key words: gsΛ $T_{0,}$ gsΛ $T_{1,}$ gsΛ $T_{2,}$ gsΛ Urysohn ; gsΛ - compact

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INTRODUCTION

A certain number of separation axioms for topological spaces are provided, all of which are good extensions of the topological (T_0,), (T_1), or (T_2) spaces. All valid implications between the different axioms are studied and counterexamples are given for the non valid ones.

In this paper some axioms related to gsΛ sets are established and some properties are examined. It highlights the definition and properties of some of the separation axioms like gsΛ-Ti ,i=0,1,2 and their inter relationships with other known separation axioms are studied.

Let us define some separation axioms as follows

Definition:1

Let (X, τ) be a topological space

1. [6] a subset A of space X is called λ-closed if

   A = B∩C, where B is a Λ-set and C is a closed set.

2. [12] a subset A of X is said to be a gsΛ closed set if $Cl_\lambda(A) \subseteq U$ whenever $A \subseteq U$, where U is semi open in X.

3. [13] a subset A of X is said to be a M.gsΛ closed map (M.gsΛ open map) if the image of each gsΛ closed set (gsΛ open set) in X is gsΛ closed (gsΛ open) in Y.

The complement of above closed sets is called its respective open sets.

The gsΛ closure (respectively closure, λ closure) of a subset A of X denoted by gsΛCl(A),(Cl(A),Cl_λ(A)is the intersection of all gsΛ closed sets (closed sets, λ closed sets) containing A.

The gsΛ interior (respectively interior, λinterior) of a subset A of X denoted by gsΛ Int(A),(int(A), λ-int(A)) is the union of all gsΛ open sets (open sets, λ open sets) containing A.

Definition :2

A topological space X is said to be

1. [8] λ T_0 (resp λ – T_1) if for x , y ∈ X such that x ≠ y there exist a λ - open set U of X containing x but not y or (resp and) a λ -open set V of X containing y but not x.

2. [8] λ-T_2 if for x,y ∈ X such that x ≠ y there exist a λ- open set U of X containing x and a λ-open set V of X containing y such that U ∩ V ≠ ∅.

Definition :3

A topological space X is said to be

1. gsΛ-T_0 (resp gsΛ-T_1) if for x ,y ∈ X such that x ≠ y there exist a gsΛ open set U of X containing x but not y or (resp and) a gsΛ open set V of X containing y but not x.
2. gsΛ T_2 if for x , y ∈ X such that x ≠ y there exist a gsΛ open set U of X containing x and a gsΛ open set V of X containing y such that U ∩ V=∅.
3. gsΛ Urysohn if for x , y ∈ X such that x ≠ y there exist a gsΛ open set U of X containing x and a gsΛ open set V of X containing y such that gsΛ Cl(U)∩gsΛ Cl(V) = ∅

Theorem :1

A space X is gsΛ T_0 if and only if for each pair of distinct points x,y of X, gsΛ Cl({x}) ≠ gsΛCl({y}).

Proof:

Sufficiency:

Suppose that x,y ∈ X, x ≠ y and gsΛ Cl({x}) ≠ gsΛ Cl({y).

Let z ∈ gsΛ Cl({x}) and z ∉ gsΛ Cl({y). We claim that x ∉ gsΛ Cl({y}).

For if x ∈ gsΛ Cl({y}) then gsΛ Cl({x) ⊆ gsΛ Cl({y}). This contradicts that

z ∉ gsΛ Cl({y}). Consequently x ∈ gsΛ Cl({y}^{C}, which is gsΛ open to which y does not belong. Thus X is gsΛ T_0 space.

Necessity:

Let X be gsΛ T_0 space and x,y ∈ X such that x ≠ y. Then there exist a gsΛ open set G containing x or y say x but not y. Then G^C is a gsΛ closed set which does not contains x but contains y. Since gsΛ Cl({y}) is the smallest gsΛ closed set containing y, gsΛCl({y})⊆ G^c and so x ∉ gsΛCl({y}). Consequently gsΛCl({x}) ≠ gsΛCl({y}).

Theorem:2

For a topological space (X,τ), the following are equivalent:

1. (X,τ) is gsΛ -T_2 .
2. If x ∈ (X,τ), then for each y ≠ x, there is a gsΛ Open set U containing x such that y ∉ gsΛ Cl(U).

Proof:

1 ⟶ 2:

Let x ∈ X. Since (X,τ)) is gsΛ -T_2 for each y ≠ x, there exist a gsΛ open sets A and B such that

x ∈ A and y ∈ B and A ∩ B = ∅ . Then x ∈ A ⊆ X \ B. Let X \ B = F. Since B is gsΛ open set, F is gsΛ closed set, where x ∈ A ⊆ F and y ∉ F. This implies y ∉ ∩ {F / F is gsʌ closed and A ⊆ F= gsʌCl(A).

2 ⟶ 1

Let x,y∈ X and x ≠ y. By (2), there exist a gsΛ open set U containing x such that

y ∉ gsΛ Cl(U). Therefore y ϵ X \ gsΛ Cl(U), where X \ gsΛ Cl(U) is gsΛ open set and

x∉ X \ gsΛ Cl(U). Also U ∩ X \ gsΛ Cl(U)= ∅. Hence (X,τ) is gsΛ T_2 .

Theorem:3

If singletons of a space X are gsΛ closed then X is gsΛ T_1 .

Proof:

Let x,y ∈ X, with x ≠ y. Now x ≠ y implies y ∈ $\{x\}^C$. Hence $\{x\}^C$ is a gsΛ open set containing y but not x. Similarly $\{y\}^C$ is a gsΛ open set containing x but not y. Thus X is gsΛ -T_1.

Theorem:4

A λ T_1 space is gsΛ -T_1 .

Proof:

A topological space X is λ T_1 if and only if the singletons of X are λ closed sets which implies that singletons of X are gsΛ closed sets. Thus by theorem-3 X is gsΛ T_1 .

Remark:

As every open set is λ open and every λ open set is gsΛ open set it is clear to observe that

1. T_0 ⟶ λT_0 ⟶ gsΛ T_0
2. T_1 ⟶ λT_1 ⟶ gsΛ T_1
3. T_2 ⟶ λ T_2 ⟶ gsΛ T_2

Reverse implication need not be true as seen from the following example.

Example:1

(X, τ) = λ O(X, τ) ={ϕ.{a},{a,b},{c,d,e},{a,c,d,e},X}

gsΛ O(X, τ) = discrete space of (X, τ) . It can be clearly observed that (X, τ) is gsΛ T_0, gsΛ T_1, gsΛ T_2, but not λ T_0. not λ T_1, not λT_2 and not T_0, not T_1, not T_2

Theorem:5

A gsΛ Urysohn space is gsΛ T_2.

Proof:

Let x and y be two distinct points of Y. Since X is gsΛ Urysohn there exist a gsΛ open set U containing x, a gsΛ open set V containing y such that gsΛ Cl(U) ∩ gsΛCl(V) implies U ∩ V= ∅. Thus X is gsΛ T_2.

Remark:1

It is easy for the readers to observe from the definitions that

gsΛ Urysohn ⟶ gsΛ T_2 ⟶ gsΛ T_1 ⟶ gsΛ T_0 .

Definition:4

A function (X,τ) ⟶ (Y,σ) is said to be

1. gsΛ irresolute if for each x ∈ X and each V in gsΛ O(Y,f(x)), there exists U ∈ gsΛ O(X,x) such that f(U) ⊆ V. Equivalently if the inverse image of each gsΛ open set in Y is gsΛ open in X
2. Quasi gsΛ irresolute if for each x ∈X and each V ∈ gsΛ O(Y,f(x)), there exists

   U ∈ gsΛ O(X,x)such that f(U)⊆ gsΛ Cl(V).

Theorem:6

Every gsΛ irresolute function f:X ⟶ Y is quasi gsΛ irresolute.

Proof is very clear as for any set V ⊆ gsΛ Cl(V).

Theorem:7

If Y is gsΛ T_2 and f:X ⟶ Y is gsΛ irresolute injection then X is gsΛ T_2 .

Proof:

Since f is injective, for any pair of distinct points x,y ∈ X, f(x) ≠ fy. As Y is gsΛ T_2 there exists U in gsΛ O(Y,f(x)) and V in gsΛ O(Y,f(y)) such that U ∩ V= ∅ . Hence f_1 U ∩ f_1 V= ∅ . Since f is gsΛ irresolute, there exists U_1 ∈ gsΛ O(X,x) and V_1 ∈ gsΛ O(X,y) such that f(U_1)⊆ U and f(V_1)⊆ V. It follows that U_1 ⊆ f_1 (U) and V_1 ⊆ f_1(V). Hence we get U_1 ∩ V_1 ⊆f_1(U)∩ f_1(V)= ∅ . Thus X is gsΛ T_2 .

Theorem:8

If Y is gsΛ Urysohn and f: X ⟶ Y is quasi gsΛ irresolute injection then X is gsΛ -T_2 .

Proof:

Since f is injective, for any pair of distinct points x,y ∈ X, f(x) ≠ f(y). As Y is gsΛ Urysohn there exists U ∈ gsΛ O(Y,f(x)) and V ∈ gsΛ O(Y,f(y)) such that gsΛ Cl(U) ∩ gsΛ Cl(V)= ∅ . Hence f^{-1} (gsΛ Cl(U)) ∩ f^{-1} (gsΛ Cl(V))= ∅ . Since f is quasi gsΛ irresolute, there exists U_1 ∈ gsΛ O(X,x) and V_1∈ gsΛ O(X,y) such that f(U_1) ⊆ gsΛ Cl(U) and f(V_1) ⊆ gsΛ Cl(V).

It follows that U_1 ⊆ f^{-1} (gsΛ Cl(U)) and V_1 ⊆ f^{-1} (gsΛ Cl(V)).

Hence we get U_1∩V_1 ⊆ f^{-1} (gsΛ Cl(U)) ∩ f^{-1} (gsΛ Cl(V)) = ∅ . Thus X is gsΛ T_2 .

On gsΛ -Compactness

Definition:5

A Collection {A_i : i ∈ Ω} of gsΛ - open sets in a topological space X is called a gsΛ - open cover of a subset S of X if S ⊆ ∪{A_i : i ∈ Ω} holds.

Definition:6

A topological space X is said to be gsΛ - Compact if every gsΛ - open cover of X has a finite sub cover.

Definition:7

A subset S of a topological space X is said to be gsΛ - Compact relative to X, if for every collection {A_i : i ∈ Ω} of gsΛ - open subsets of X such that S ⊆ ∪{A_i : i ∈ Ω} there exists a finite gsΛ - open subsets Ω_0 of Ω such that S ⊆ ∪{A_i : i ∈ Ω_0} .

Definition:8

A subset S of a topological space X is said to be gsΛ - compact if S is gsΛ -compact as a subspace of X.

Theorem:9

A gsΛ - closed subset of a gsΛ-compact space X is gsΛ - compact relative to X.

Proof:

Let A be a gsΛ - closed subset of a gsΛ - compact space X. Then X – A is gsΛ - open. Let Ω be a gsΛ -open cover for A. Then {Ω ,(X–A)} is a gsΛ - open cover for X. Since X is gsΛ - compact, it has a finite sub cover say {P_1 , P_2 , P_3 ,... P_n }= Ω_1 . If X– A $\notin$ Ω_1 then Ω_1 is a finite sub cover of A. If (X– A) $\in$ Ω_1, then Ω_1 – (X – A) is a sub cover of A. Thus A is gsΛ -compact relative to X.

Theorem:10

For a topological space (X, τ), the following are equivalent:

1) (X, τ) is gsΛ - Compact.
2) Every proper gsΛ -closed set is gsΛ - compact relative to X.

Proof:

(1)$\rightarrow$ (2): follows from Theorem 6.2.5

(2) $\rightarrow$ (1):

Let {Vα : α $\in$ I }be a gsΛ - open cover of X. Then X $\subseteq\cup${ Vα : $\alpha \in$ I}. We choose and fix α_0 $\in$ I.

Then X – Vα_0 is a proper gsΛ -closed subset of X and X - Vα_0 $\subseteq$ $\cup${ Vα : $\alpha \in$ I–$\alpha_{0\}}$. Therefore

{Vα : α $\in$ I – $\alpha_{0\}}$ is a gsΛ- open cover of X – Vα_0. By hypothesis, there exist a finite subset I_0 of { I – $\alpha_{0\}}$ such that X – Vα_0 $\subseteq$ $\cup${ Vα : $\alpha \in I_0$ }.

Therefore X $\subseteq$ $\cup${ Vα : $\alpha \in I_0 \cup \{\alpha_0\}$} . Hence X is gs$\Lambda$ - Compact.

Theorem:11

Let f: (X, τ) $\rightarrow$ (Y, σ) be a surjective, gsΛ - continuous map. If X is gsΛ - compact then Y is compact.

Proof:

Let {A_i : i $\in$ I} be an open cover of Y. Then f^{-1} (A_i): i $\in$ I is a gsΛ - open cover of X. Since X is gsΛ - compact, it has a finite sub cover say $f^{-1}(A_1)$,$f^{-1}(A_2)$,....$f^{-1}(A_n)$. Since f is surjective A_1 ,A_2 ,A_3 ,...An is an open cover of Y. Thus Y is compact.

Theorem:12

Let f:(X, τ) $\rightarrow$ (Y, σ) be a surjective, gsΛ - irresolute map. If X is gsΛ - compact then Y is gsΛ - compact.

Proof:

Let A_i : i $\in$ Ω be a gsΛ- open cover of Y. Then f^{-1} (A_i): I $\in$ Ω is a gsΛ-open cover of X as f is gsΛ - irresolute. Since X is gsΛ - compact, it has a finite sub cover say f^{-1} (A_1) , $f^{-1}(A_2)$,.... f^{-1} (A_n) . Since f is surjective {A_1 ,A_2 ,A_3 ,...A_n } is an gsΛ - open cover of Y. Thus Y is gsΛ - compact.

Theorem:13

If a map f:(X, τ) $\rightarrow$ (Y, σ) is gsΛ - irresolute map and a subset S of X is gsΛ - compact relative to X, then the image f(S) is gsΛ - compact relative to Y.

Proof:

Let Ai :i $\in$ Ω be a collection of gsΛ - open sets in Y such that f(S) $\subseteq\cup${ A_i : i $\in$ Ω. Then S $\subseteq${ f^{-1} (A_i) : i $\in$ Ω}, where f^{-1} (A_i) is a gsΛ - open cover of X for each i. Since S is gsΛ - compact relative to X, there exist a finite sub collection $f^{-1}(A_1)$, $f^{-1}(A_2)$,.... f^{-1} (A_n) such that S$\subseteq\cup${ f^{-1} (A_i),i=1 to n}. That is f(S) $\subseteq\cup${A i : i=1to n}. Hence f(S) is gsΛ - compact relative to Y.

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