ISSN 2347-1921
Volume 12 Number 05
Journal of Advances in Mathematics

Some properties of the superior and inferior semi inner product function associated to the 2-norm

Stela ÇENO
Department of Mathematics. Faculty of Natural Science.
"A.Xhuvani" University of Elbasan. Albania
stelaceno85@gmail.com

ABSTRACT

Special properties that the scalar product enjoys and its close link with the norm function have raised the interest of researchers from a very long period of time.

S.S. Dragomir presents concrete generalizations of the scalar product functions in a normed space and deals with the interesting properties of them.

Based on S.S. Dragomir's idea in this paper we treat generalizations of superior and inferior scalar product functions in the case of semi-normed spaces and 2-normed spaces.

Keywords

Semi-norm; 2-norm; superior (inferior) semi scalar product function; superior (inferior) scalar product function associated to the 2-norm.

Academic Discipline And Sub-Disciplines

Mathematics; Functional Analysis

1. INTRODUCTION

Before explaining the main results of this paper, we introduce some common known concepts.

Definition 1.1 Let X be a complex (real) vector space. We shall say that a complex (real) semi inner product is defined on X , if to any $x, y \in X$ there corresponds a complex (real) number (x, y) and the following properties hold:

1. $(x+y,z)=(x,z)+(y,z)$

   $(\lambda x, y)=\lambda(x, y)$ for $x, y, z \in X$, λ complex (real)

2. $(x,x)>0$ for $x \neq 0$

3. $|(x,y)|^2 \leq (x,x)(y,y)$

We then call X a complex (real) semi inner product space.

Definition 1.2: A semi norm is a function on vector space X , denoted $p(x)$ such that the following conditions hold:

1. $p(x) \geq 0$

2. $p(\lambda x)=|\lambda| p(x) \quad \forall \lambda \in K$

3. $p(x+y) \leq p(x)+p(y) \qquad \forall x, y \in X$.

Definition 1.3: Let X be a real linear space of dimension greater than 1 and let $\|\cdot,\cdot\|$ be a real valued function on $X \times X$ satisfying the following conditions:

1. $\|x,y\|=0$ if and only if *x* and *y* are linearly dependent.

2. $\|x,y\|=\|y,x\|$

3. $\|\lambda x,y\|=|\lambda|\|x,y\|$

4. $\|x,y+\| \leq \|x,y\|+\|x,z\|$ for all $x, y, z \in X$ and $\lambda \in R$.

$\|\cdot,\cdot\|$ is called a 2-norm on X and $(X,\|\cdot,\cdot\|)$ is called a linear 2-normed space. It is easily proven that the 2-norm is non-negative.

The 2-norm functions defined for the first time by Gahler.S, represent for many authors [2],[4] an area with many generalized results of the norm function.

Definition 1.4: The Banach space X is said to have a Gateaux differentiable norm at $x_0 \in S(x)$ whenever given $y \in S(x)$: $\lim_{\lambda \to 0} \frac{\|x_0 + \lambda y\| - \|x_0\|}{\lambda}$ exists.

SUPERIOR AND INFERIOR SEMI INNER PRODUCT FUNCTION ASSOCIATED TO THE 2-NORM

G.Lumer [5] generalized for the first time the scalar product by replacing its axiomatic with much more limited conditions in relation to homogeneity, linearity, attribute of symmetry and and the same time he made the attempt to introduce the Cauchy inequality.

Other authors like R. Gilles [3], E.Torrance [7], B. Nath [6], study the connection of orthogonality and strictly convex in spaces obtained by generalized functions of the scalar product.

Let $(X,\|\cdot\|)$ be a normed linear space over the real or complex number field K. The mapping $f : X \to R$, $f(x) = \frac{1}{2}\|x\|^2$ is obviously convex and then there exists the following limits:

$$(x,y)_i = \lim_{t\to 0-} \frac{\|y+tx\|^2 - \|y\|^2}{2t} \quad (x,y)_s = \lim_{t\to 0+} \frac{\|y+tx\|^2 - \|y\|^2}{2t}$$

for every two elements in X. The mapping $(\cdot,\cdot)_s, (\cdot,\cdot)_i$ will be called the superior (inferior) semi inner product associated to the norm $\|\cdot\|$.

This functions are provided by S.S.Dragomir in [1], and he presents some concrete properties of these functions.

Definition 2.1: Let L be a vector space and $p : L \to R^+$ a semi norm function in L. There exists the following limits:

$$(x,y)_i = \lim_{t\to 0-} \frac{p^2(y+tx) - p^2(y)}{2t} \text{ and } (x,y)_s = \lim_{t\to 0^+} \frac{p^2(y+tx) - p^2(y)}{2t}$$

The mapping $(x,y)_i$, $(x,y)_s$ will be called the inferior (superior) semi-inner product associated to the semi-normed $p(x)$.

Proposition 2.1: Let X be a linear space and $p(x)$ a semi norm in this space. Then, the following statements are true.

a) $(x,x)_p = p^2(x)$ for all $x \in X$.

b) $(ix,x)_p = (x,ix)_p = 0$ for all $x \in X$.

c) $(\lambda x, y)_p = \lambda (x,y)_p$ for all nonnegative scalar λ and $x, y \in X$.

d) $(x, \lambda y)_p = \lambda (x,y)_p$ for all nonnegative scalar λ and $x, y \in X$.

e) $(\lambda x, y)_p = \lambda (x,y)_q$ if $\lambda < 0$ and $x, y \in X$.

f) $(x,\lambda y)_p = \lambda(x,y)_q$ if $\lambda < 0$ and $x, y \in X$.

g) $(ix,y)_p = -(x,iy)_q$ for all $x, y \in X$; where $p,q \in \{s,i\}$ and $p \neq q$.

Proof: The proof is as follows:

a) $(x,x)_p = \lim_{t\to\pm 0} \frac{p^2(x+tx)-p^2(x)}{2t} = p^2(x)\lim_{t\to\pm 0}\frac{|1+t|-1}{t} = p^2(x)$ for all $x \in X$.

b) It is clear that: $(ix,x)_p = (x,ix)_p = \lim_{t\to\pm 0}\frac{p^2(ix+tx)-p^2(ix)}{2t}$

$$= p^2(x)\lim_{t\to\pm 0}\frac{|i+t|^2-1}{t} = p^2(x)\lim_{t\to\pm 0}\frac{\sqrt{1+t^2}-1}{2t} = 0$$

for all $x \in X$.

c) For all $x \in X$ we have:

$(\lambda x,y)_p = \lim_{t\to\pm 0}\frac{p^2(y+\lambda tx)-p^2(y)}{2t}$. Denoting $u = \lambda t$,we have:

$$(\lambda x,y)_p = \begin{cases} \lambda \lim_{u\to\pm 0}\frac{p^2(y+ut)-p^2(y)}{2u} & \lambda \geq 0 \\ \lambda \lim_{u\to\mp 0}\frac{p^2(y+ut)-p^2(y)}{2u} & \lambda < 0 \end{cases} = \begin{cases} \lambda(x,y)_p \lambda \geq 0 \\ \lambda(x,y)_q \lambda < 0 \end{cases}.$$

The proof of the statements d,e,f go likewise:

h) We have:

$$(ix,y)_p = \lim_{t\to\pm 0}\frac{p^2(ix+tx)-p^2(ix)}{2t} = \lim_{t\to\pm 0}\frac{p^2(iy-tx)-p^2(iy)}{2t}$$

$= (x,-iy)_p = -(x,iy)_q$ for all $x, y \in X$.

Corollary 1: With the above assumptions, we have:

$(\alpha x,\beta y)_p = \alpha\beta(x,y)_p$ for all $\alpha,\beta \in R$ with $\alpha\beta \geq 0$ and $x, y \in X$.

Corollary 2: We also have: $(-x,y)_p = (x,-y)_p = -(x,y)_q$

where $x, y \in X$ $p,q \in \{s,i\}$ and $p \neq q$.

In this paper we present some properties of the superior and inferior semi inner product associated to the 2-norm.

First we will define the inferior and superior semi inner product associated to the 2-norm function. The analogue functions for a fixed point $b \in X$ are as follows:

$$(x,y)_s^b = \lim_{t\to 0+}\frac{\|y+tx,b\|^2-\|y,b\|^2}{2t} \text{ and } (x,y)_i^b = \lim_{t\to 0-}\frac{\|y+tx,b\|^2-\|y,b\|^2}{2t}$$

Proposition 3.1: Let X be a vectorial space and $\|x,y\|$ a 2- norm function in X . Let $(x,y)_s^b$ and $(x,y)_i^b$ be the superior (inferior) semi-inner product associated to the 2-norm. Then, the following statements are true:

a) $(x,x)_p^b = \|x,b\|^2$ for all $x \in X$ and $b \in X$ (b is a fixed point).

b) $(ix,x)_p = (x,ix)_p = 0$ for all $x \in X$ and $b \in X$.

c) $(\lambda x,y)_p = \lambda (x,y)_p$ for all nonnegative scalar λ and $x,y \in X$ and $b \in X$.

d) $(x,\lambda y)_p = \lambda (x,y)_p$ for all nonnegative scalar λ and $x,y \in X$ and $b \in X$.

e) $(\lambda x,y)_p = \lambda (x,y)_q$ if $\lambda < 0$ and $x,y \in X$ and $b \in X$.

f) $(x,\lambda y)_p = \lambda (x,y)_q$ if $\lambda < 0$ and $x,y \in X$ and $b \in X$.

g) $(ix,y)_p = -(x,iy)_q$ for all $x,y \in X$ and $b \in X$,where $p,q \in \{s,i\}$ and $p \neq q$.

Proof:

a) $$(x,x)_p^b = \lim_{t\to 0\pm} \frac{\|x+tx,b\|^2 - \|x,b\|^2}{2t} = \lim_{t\to 0\pm} \frac{\|x(1+t),b\|^2 - \|x,b\|^2}{2t}$$

$$= \lim_{t\to 0\pm} \frac{|1+t|^2 \|x,b\|^2 - \|x,b\|^2}{2t} = \|x,b\|^2 \lim_{t\to 0\pm} \frac{|1+t|^2 - 1}{2t}$$

$$= \|x,b\|^2 .$$

b) $$(ix,x)_p^b = (x,ix)_p^b = \lim_{t\to 0\pm} \frac{\|ix+tx,b\|^2 - \|ix,b\|^2}{2t}$$

$$= \lim_{t\to 0\pm} \frac{\|x(i+t),b\|^2 - \|ix,b\|^2}{2t}$$

$$= \|x,b\|^2 \lim_{t\to 0\pm} \frac{|i+t|^2 - 1}{2t} = \|x,b\|^2 \cdot 0 = 0 .$$

c) and e)

$(\lambda x,y)_p^b = \lim_{t\to 0\pm} \dfrac{\|y+\lambda tx,b\|^2 - \|y,b\|^2}{2t}$. Denoting $u = \lambda t$, we have:

$$(\lambda x,y)_p^b = \lim_{u\to 0\pm} \lambda \frac{\|y+ux,b\|^2 - \|y,b\|^2}{2u} =$$

$$= \begin{cases} \lambda \lim_{u\to 0\pm} \dfrac{\|y+ux,b\|^2 - \|y,b\|^2}{2u} & \text{for } \lambda \geq 0 \\ \lambda \lim_{u\to 0\mp} \dfrac{\|y+ux,b\|^2 - \|y,b\|^2}{2u} & \text{for } \lambda < 0 \end{cases} = \begin{cases} \lambda (x,y)_p^b & \text{for } \lambda \geq 0 \\ \lambda (x,y)_q^b & \text{for } \lambda < 0 \end{cases}.$$

The proofs of the statements d,e,f, go likewise.

h) $(ix,y)_p^b = \lim_{t\to 0\pm} \dfrac{\|y+itx,b\|^2 - \|y,b\|^2}{2t} =$

$$= \lim_{t\to 0\pm} \frac{\|iy-tx,b\|^2 - \|iy,b\|^2}{2t} = (x,-iy)_p^b = -(x,iy)_q^b .$$

Proposition 3.2: Let $(X,\|\cdot,\cdot\|)$ be a 2-normed space. Than one has:

(i) The following inequality is valid:

$$\frac{\|x+ty,b\|^2 - \|x,b\|^2}{2t} \geq (y,x)_s^b \geq (y,x)_i^b \geq \frac{\|x+sy,b\|^2 - \|x,b\|^2}{2s}$$

for all $x,y,\in X$ and $t>0, s<0$.

(ii) $\left|(x,p)_p^b\right| \leq \|x,b\|\cdot\|y,b\|$ for all $x,y,\in X$

(iii) The mapping $(\cdot,\cdot)_s^b\left((\cdot,\cdot)_i^b\right)$ is sub(super)-additive in the first variable i.e, for $x_1,x_2,y,\in X$

$(x_1+x_2,y)_{s(i)}^b \leq (\geq)(x_1,y)_{s(i)}^b + (x_2,y)_{s(i)}^b$ hold.

Proof: (i) Let us consider the mapping $g:[0,\infty)\to R, g(t)=\frac{1}{2}\|x+ty,b\|^2$ for x,y,b fixed in X. It is clear that $g(t)$ is convex on $[0,\infty)$ and then:

$\dfrac{g(t)-g(0)}{t-0} \geq g'_+(0)$ for $t>0$ which means that:

$$\frac{\|x+ty,b\|^2 - \|x,b\|^2}{2t} \geq \lim_{t\to 0+} \frac{\|x+ty,b\|^2 - \|x,b\|^2}{2t} = (y,x)_s^b .$$

The second inequality follows by the fact that: $g'_+(0) \geq g'_-(0)$ if $g(t)$ is any convex mapping of a real variable. The last fact is also obvious.

(ii) Let $x,y\in X$. Then : $\left|(x,y)_p^b\right| = \left|\lim_{t\to 0\pm} \dfrac{\|y+tx,b\|^2 - \|y,b\|^2}{2t}\right|$

$$= \left|\lim_{t\to 0\pm} \frac{\|y+tx,b\| + \|y,b\|}{2t}\right| \left|\lim_{t\to 0\pm} \frac{\|y+tx,b\| - \|y,b\|}{2t}\right|$$

$\leq \|y,b\| \lim_{t\to \pm 0} \dfrac{\|y,b\| + \|tx,b\| - \|y,b\|}{|t|} = \|y,b\|\cdot\|x,b\|$ and the statement is proved.

(iii)By the usual properties of the 2-norm one has:

$$(x_1+x_2,y)^b_{s(i)} = \frac{1}{2}\|2y,b\| \lim_{t\to\pm 0} \frac{\|y+t(x_1+x_2),b\|-\|2y,b\|}{t}$$

$$= \|y,b\| \lim_{t\to\pm 0} \frac{\|y+tx_1+y+tx_2,b\|-2\|y,b\|}{t}$$

$$\le(\ge)\|y,b\| \lim_{t\to\pm 0} \frac{\|y+tx_1,b\|+\|y+tx_2,b\|-2\|y,b\|}{t}$$

$$= \|y,b\| \lim_{t\to\pm 0} \frac{\|y+tx_1,b\|-\|y,b\|}{t} + \|y,b\| \lim_{t\to\pm 0} \frac{\|y+tx_2,b\|-\|y,b\|}{t}$$

$$= (x_1,y)^b_{s(i)} + (x_2,y)^b_{s(i)} \qquad \text{for every } x_1,x_2,y\in X .$$

Now we introduce the Gateaux differentiable 2-norm for a fixed point $b\in X$ by:

Definition 3.1: The Banach space X is said to have a Gateaux differentiable 2-norm at $x_0\in S(x)$ and for a fixed point $b\in X$ whenever given $y\in S(x)$: $\lim_{\lambda\to 0}\frac{\|x_0+\lambda y,b\|-\|x_0,b\|}{\lambda}$ exists.

Theorem 3.1: Let $(X,\|\cdot,\cdot\|)$ be a 2-normed space. Then the following statements are equivalent:

a) The 2-norm is Gateaux differentiable on $X\setminus\{0\}$,i.e,the space is smoth.

b) The semi inner product $(\cdot,\cdot)^b_p$ is homogeneous in the second argument.

c) The semi inner product $(\cdot,\cdot)^b_p$ is homogeneous in the first argument.

d) The semi inner product $(\cdot,\cdot)^b_p$ is linear in the first argument.

where $p=s, p=i$.

Proof: The proof is for the case $p=s$. The case $p=i$ goes equally.

a) $\Rightarrow$b) Since $(\cdot,\cdot)^b_s$ is positive homogeneous in the second argument it is sufficient to show that: $(x,-y)^b_s = -(x,y)^b_s$ for all $x,y\in X$.

The Gateaux differentiable of the 2-norm implies that:

$$(x,-y)^b_s = \lim_{t\to 0}\frac{\|(-y)+tx,b\|^2-\|-y,b\|^2}{2t} = \lim_{t\to 0}\frac{\|y-tx,b\|^2-\|y,b\|^2}{2t}$$

$$= \lim_{t\to 0}\frac{\|y+(-t)x,b\|^2-\|y,b\|^2}{2t} = -\lim_{t\to 0}\frac{\|y+sx,b\|^2-\|y,b\|^2}{2s}$$

$=-(x,y)_s^b$ and the implication is proved.

b) $\Rightarrow$ c) We will show that :

$$(-x,y)_s^b = -(x,y)_s^b \text{ for all } x,y \in X .$$

Indeed, since: $(-x,y)_s^b = (x,-y)_s^b = -(x,y)_s^b$ for all $x,y \in X$, and the proof of the statement is completed.

c) $\Rightarrow$ d) Since $(\cdot,\cdot)_s^b$ is subaddite (see proposition 3.2) and homogeneous, it is linear in the first argument.

d) $\Rightarrow$ a) Let $x,y \in X$ with $y \neq 0$. Then:

$$\lim_{t\to 0+} \frac{\|y+tx,b\|-\|y,b\|}{t} = \frac{(x,y)_s^b}{\|y\|} = -\frac{(-x,y)_s^b}{\|y\|}$$

$$=-\lim_{t\to 0+} \frac{\|y+(-t)x,b\|-\|y,b\|}{t} = \lim_{s\to 0-} \frac{\|y+sx,b\|-\|y,b\|}{s}$$

i.e, the $\|\cdot,\cdot\|$ is Gateaux differentiable on $X\setminus\{0\}$ and the theorem is thus proved.

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